13.3 Measurements on Curves in 3D

Today: *Unit Tangent and Normal, Arc Length, and Curvature*

Entry Task

Consider $\boldsymbol{r}(t) = \langle t, t^2, 5 \rangle$.

Find the unit tangent vector **T**(t).

Thm: $\boldsymbol{T}$ and $\boldsymbol{T}'$ are always orthogonal.

Proof: Since $\boldsymbol{T} \cdot \boldsymbol{T} = |\boldsymbol{T}|^2 = 1$,

we differentiate both sides to get

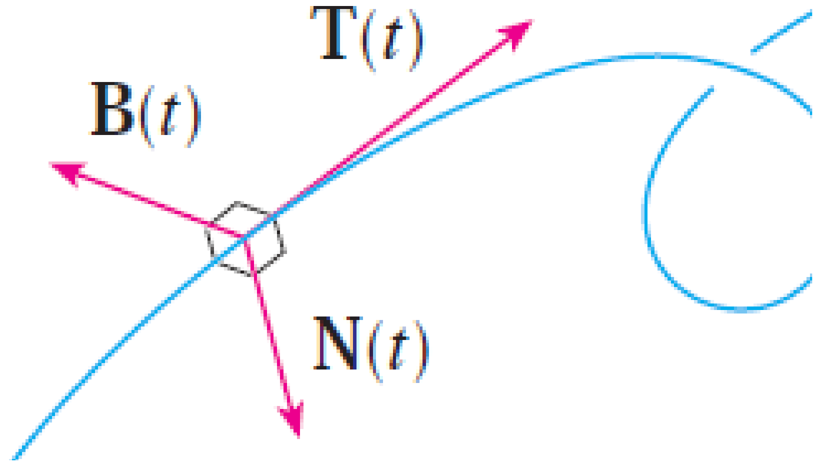
$$\boldsymbol{T}' \cdot \boldsymbol{T} + \boldsymbol{T} \cdot \boldsymbol{T}' = 0.$$

So $2\boldsymbol{T} \cdot \boldsymbol{T}' = 0$.

Thus, $\boldsymbol{T} \cdot \boldsymbol{T}' = 0$. (QED)

$$\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{r}}'(t)}{|\vec{\boldsymbol{r}}'(t)|} = \text{unit tangent}$$

$$\vec{\boldsymbol{N}}(t) = \frac{\vec{\boldsymbol{T}}'(t)}{|\vec{\boldsymbol{T}}'(t)|} = \text{principal unit normal}$$



TNB-Frame Facts:

- $\vec{\boldsymbol{T}}(t)$ and $\vec{\boldsymbol{N}}(t)$ point in the tangent and *inward* directions, respectively. Together they give a good approximation of the "plane of motion" called the *osculating (kissing) plane.*
- $\vec{\boldsymbol{T}}(t)$, $\vec{\boldsymbol{N}}(t)$, $\vec{\boldsymbol{r}}'(t)$, and $\vec{\boldsymbol{r}}''(t)$ are ALL parallel to the osculating plane.
  If we also define
  $\vec{\boldsymbol{B}}(t) = \vec{\boldsymbol{T}}(t) \times \vec{\boldsymbol{N}}(t) = \text{binormal}$
  Then $\vec{\boldsymbol{B}}(t)$ is orthogonal to all of $\vec{\boldsymbol{T}}(t)$, $\vec{\boldsymbol{N}}(t)$, $\vec{\boldsymbol{r}}'(t)$, and $\vec{\boldsymbol{r}}''(t)$.

Distance Traveled on a Curve

The dist. traveled along a curve from
t = *a* to *t* = *b* is

$$\int_a^b |\boldsymbol{r}'(t)|dt = \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}dt$$

Note: 2D is same without the $z'(t)$.
We derived this in Math 125.

Examples:

1. Find the length of the curve
$$\boldsymbol{r}(t) = \langle 1 + 3\text{t}, 2\text{t}, 5t\rangle$$
from $t = 0$ to $t = 2$.

2. Find the length of the curve
$\boldsymbol{r}(t) = \langle \cos(2t), \sin(2t), 2\ln(\cos(t)) \rangle$
from $t = 0$ to $t = \pi/3$.

If the curve is "traversed once" we call this distance the **arc length**.

Example: $x = \cos(t), y = \sin(t)$

(a) Find the distance traveled by this object from $t = 0$ to $t = 6\pi$.

(b) Find the arc length of the path over which this object is traveling.

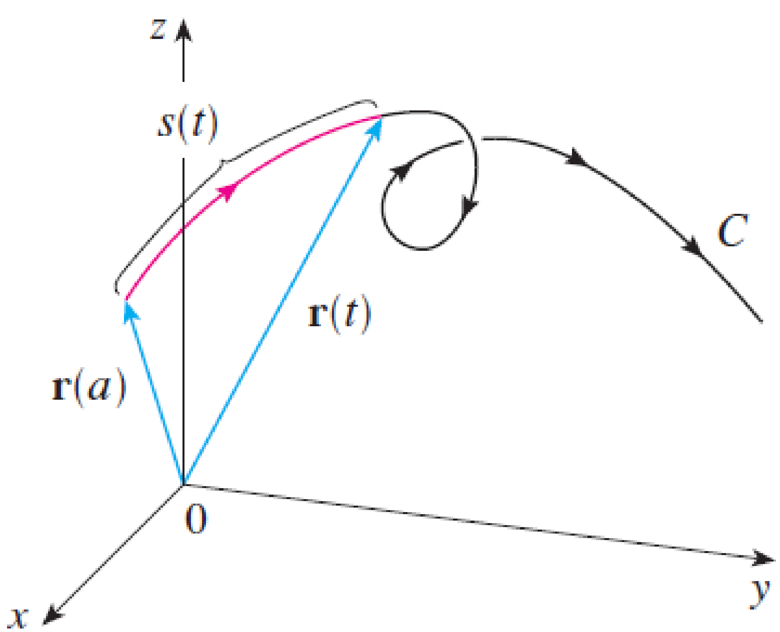
Arc Length Function

The distance from time *u = a* to *u = t* is called the *arc length function*

$$s(t) = \int_a^t |\vec{\boldsymbol{r}}'(u)|du = \text{distance}$$

Note:

$$\frac{ds}{dt} = |\vec{\boldsymbol{r}}'(t)| = \text{speed}$$



Example: x = 3 + 2t, y = 4 – 5t

(a) Find the arc length function (from 0 to t).

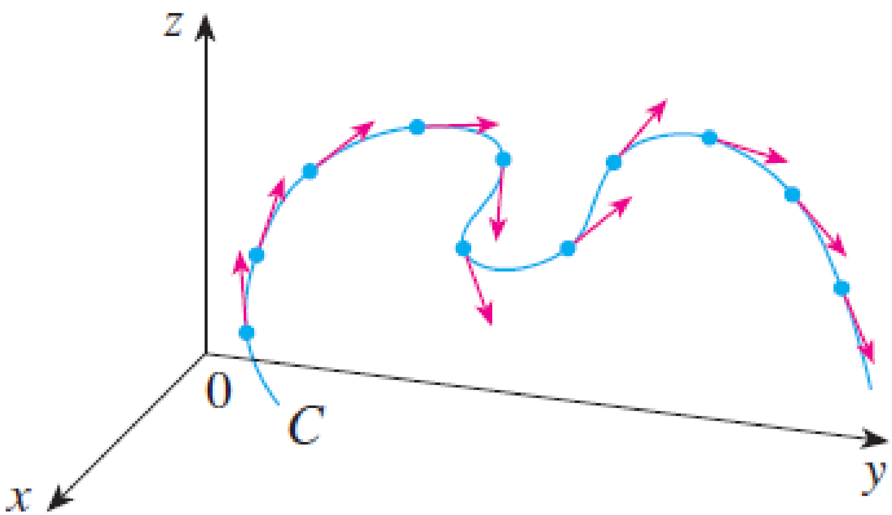
(b) ***Reparameterize*** in terms of *s(t)*.

Curvature

The **curvature** at a point, *K*, is a measure of how quickly a curve is changing direction at that point.

That is, we define

$$K = \frac{change\ in\ direction}{change\ in\ distance}$$



Roughly, how much does your direction change if you move a small amount ("one inch") along the curve?

$$K \approx \left|\frac{\vec{T_2} - \vec{T_1}}{\text{"one inch"}}\right| = \left|\frac{\Delta\vec{T}}{\Delta s}\right|$$

So we define:

$$K = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right|$$

Computation

$$K = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right|$$

is not easy to compute directly, so we derive some *shortcuts*

1st shortcut:

$$K(t) = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \left|\frac{d\vec{\boldsymbol{T}}/dt}{ds/dt}\right| = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|}$$

2nd shortcut

$$K(t) = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|} = \frac{|\vec{\boldsymbol{r}}'(t) \times \vec{\boldsymbol{r}}''(t)|}{|\vec{\boldsymbol{r}}'(t)|^3}$$

Example: Find the curvature function for $\boldsymbol{r}(t) = \langle t, \cos(2t), \sin(2t)\rangle$.

Answer:

$$\boldsymbol{r}'(t) = \langle 1, -2\sin(2t), 2\cos(2t)\rangle$$
$$\boldsymbol{r}''(t) = \langle 0, -4\cos(2t), 4\sin(2t)\rangle$$

$|\boldsymbol{r}'(t)| = \sqrt{1 + 4\sin^2(2t) + 4\cos^2(2t)}$

so $|\boldsymbol{r}'(t)| = \sqrt{5}$

$\boldsymbol{r}'(t) \times \boldsymbol{r}''(t) = \langle -8, -4\sin(2t), -4\cos(2t)\rangle$

So $|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)| = \sqrt{64 + 16} = \sqrt{80}$

$$\frac{|\vec{\boldsymbol{r}}'(t) \times \vec{\boldsymbol{r}}''(t)|}{|\vec{\boldsymbol{r}}'(t)|^3} = \frac{\sqrt{80}}{\sqrt{5}^3} = \sqrt{\frac{80}{125}} = 0.8$$

This curve has constant curvature.

Aside: The *radius of curvature* is the radius of the circle that would best fit this curve. It is always 1/K.
In this case it would be 1/0.8 = 1.25.

In other words, moving along this curve is like moving around a circle of radius 1.25, that is another way to think of how “curvy” it is)

Proof of shortcut:

Theorem: $\frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}\prime(t)|} = \frac{|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|^3}$

Proof:

Since $\boldsymbol{T}(t) = \frac{\boldsymbol{r}\prime(t)}{|\boldsymbol{r}'(t)|}$, we have

$$\boldsymbol{r}'(t) = |\boldsymbol{r}'(t)|\boldsymbol{T}(t).$$

Differentiating this gives (prod. rule):

$$\boldsymbol{r}''(t) = |\boldsymbol{r}'(t)|'\boldsymbol{T}(t) + |\boldsymbol{r}'(t)|\boldsymbol{T}'(t).$$

Take cross-prod. of both sides with $\vec{\boldsymbol{T}}$:

$$\boldsymbol{T}\times\boldsymbol{r}'' = |\boldsymbol{r}'|'\,(\boldsymbol{T}\times\boldsymbol{T}) + |\boldsymbol{r}'|\,(\boldsymbol{T}\times\boldsymbol{T}').$$

Since $\boldsymbol{T}\times\boldsymbol{T} = < 0, 0, 0 >$ (why?) and $\boldsymbol{T} = \frac{\boldsymbol{r}\prime}{|\boldsymbol{r}'|}$, we have

$$\frac{\boldsymbol{r}'\times\boldsymbol{r}''}{|\boldsymbol{r}'|} = |\boldsymbol{r}'|\,(\boldsymbol{T}\times\boldsymbol{T}').$$

Taking the magnitude gives (why?)

$$\frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|} = |\boldsymbol{r}'|\,|\boldsymbol{T}\times\boldsymbol{T}'| = |\boldsymbol{r}'|\,|\boldsymbol{T}||\boldsymbol{T}'|sin\left(\frac{\boldsymbol{\pi}}{\mathbf{2}}\right),$$

Since $|\boldsymbol{T}| = 1$, we have

$$|\boldsymbol{T}'| = \frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|^2}$$

Therefore

$$K = \left|\frac{dT}{ds}\right| = \frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|} = \frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|^3}.$$

Note: To find curvature for a 2D function, *y = f(x)*, we can form a 3D vector function as follows

$$\boldsymbol{r}(x) = \langle x, f(x), 0\rangle$$

so $\boldsymbol{r}'(x) = \langle 1, f'(x), 0\rangle$ and

$$\boldsymbol{r}''(x) = \langle 0, f''(x), 0\rangle$$

$$|\boldsymbol{r}'(x)| = \sqrt{1 + (f'(x))^2}$$

$$\boldsymbol{r}' \times \boldsymbol{r}'' = \langle 0,0, f''(x)\rangle$$

Thus,

$$K(x) = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|^3} = \frac{|f''(x)|}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}$$

Example: $f(t) = x^2$

At what point (x, y, z) is the curvature maximum?

Summary of 3D Curve Measurement Tools:

Given $\vec{\boldsymbol{r}}(t) = < x(t), y(t), z(t) >$

$\vec{\boldsymbol{r}}'(t) =$ a tangent vector

$$s(t) = \int_0^t |\vec{\boldsymbol{r}}'(t)| dt$$

$$K = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \frac{|\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|^3}$$

$$\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{r}}'(t)}{|\vec{\boldsymbol{r}}'(t)|} = \text{unit tangent}$$

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